Constrained Tangential Motion on an Expanding Circular Front

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ABSTRACT. We investigate the dynamics of point particles constrained to the boundary of a uniformly expanding circle with expansion speed c. While a fixed laboratory-frame speed constraint forbids tangential motion, we show that a velocity constraint imposed relative to the expanding front allows tangential motion with any speed up to the expansion speed. The resulting trajectories are logarithmic spirals with angular velocity decaying as $1/t$. In the extremal case of maximal tangential speed, the constraint is enforced by nonholonomic forces that decay algebraically in time. The model provides a minimal and exactly solvable example of constrained motion on time-dependent manifolds and clarifies the distinction between absolute and comoving kinematics.

I. INTRODUCTION

Constraints on moving geometries arise naturally in systems ranging from propagating wave fronts and growing interfaces to effective horizon models and active matter. A particularly simple yet subtle example is motion constrained to the boundary of an expanding domain. A common intuition suggests that tangential motion along such a boundary is incompatible with fixed-speed propagation. In this Letter, we show that this conclusion depends crucially on how the speed constraint is defined.

We consider the minimal system of a point particle constrained to the boundary of a circle expanding uniformly at speed c. We demonstrate that while a fixed laboratory-frame speed constraint indeed forbids tangential motion, a velocity constraint formulated relative to the expanding front allows nontrivial tangential dynamics. The resulting motion is analytically solvable and exhibits logarithmic spiral trajectories.

II. MODEL AND CONSTRAINTS

We work in the plane using polar coordinates (r, θ). The kinetic Lagrangian of a free particle of mass m is

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2).$$

A. Geometric Constraint

The particle is constrained to lie on the boundary of a circle expanding at constant speed c:

$$g_1(r, t) = r - ct = 0.$$

This is a holonomic, explicitly time-dependent constraint.

B. Velocity Constraint Relative to the Front

Instead of fixing the particle's speed in the laboratory frame, we impose a bound on its velocity relative to the expanding boundary:

$$g_2(\dot{r}, r, \dot{\theta}, t) = (\dot{r} - c)^2 + (r\dot{\theta})^2 \leq c^2.$$

This inequality expresses that the particle's velocity in the comoving frame of the expanding front does not exceed c. The equality corresponds to the extremal case of maximal relative motion.

This formulation differs fundamentally from a laboratory-frame speed constraint and allows tangential motion along the boundary.

III. REDUCED DYNAMICS ON THE EXPANDING BOUNDARY

Imposing the geometric constraint yields

$$r(t) = ct, \dot{r} = c.$$

The velocity constraint then reduces to

$$| r\dot{\theta} | \leq c.$$

We introduce a dimensionless parameter $\alpha \in [-1,1]$ via

$$r\dot{\theta} = \alpha c.$$

The angular equation of motion becomes

$$\dot{\theta} = \frac{\alpha}{t},$$

which integrates to

$$\theta(t) = \theta_0 + \alpha \ln \left(\frac{t}{t_0}\right),$$

where $\theta_0 = \theta(t_0)$.

IV. TRAJECTORIES

The particle trajectory in Cartesian coordinates is

$$x(t) = ct \cos\left(\theta_0 + \alpha \ln\frac{t}{t_0}\right),$$
$$y(t) = ct \sin\left(\theta_0 + \alpha \ln\frac{t}{t_0}\right).$$

Eliminating time yields

$$r(\theta) = r_0\, e^{(\theta-\theta_0)/\alpha}, r_0 = ct_0,$$

which is the equation of a logarithmic spiral for $\alpha \neq 0$.

The purely radial case is recovered for $\alpha = 0$. The tangential speed

$$v_\theta = r\dot{\theta} = \alpha c$$

is constant, while the angular velocity decays universally as $1/t$.

V. CONSTRAINT FORCES

For $|\alpha| < 1$, the inequality constraint is inactive and no tangential constraint force is required. In the extremal case $|\alpha| = 1$, the constraint is saturated and must be enforced dynamically.

Using Lagrange multipliers for the holonomic constraint and the saturated velocity constraint, one finds the constraint force

$$\mathbf{F} = F_r\, \mathbf{e}_r + F_\theta\, \mathbf{e}_\theta$$

with components

$$F_r(t) = -\frac{mc}{t}, \qquad F_\theta(t) = \frac{m\alpha c}{t}.$$

The force magnitude decays algebraically,

$$|\mathbf{F}| = \frac{mc}{t}\sqrt{1+\alpha^2},$$

indicating that the constraint becomes asymptotically weak at long times, despite maintaining constant tangential motion.

VI. DISCUSSION

Our results show that tangential motion on an expanding boundary is not kinematically forbidden but depends on the interpretation of the velocity constraint. A fixed laboratory-frame speed constraint eliminates tangential degrees of freedom, whereas a bound imposed relative to the expanding front permits a continuous family of motions interpolating between pure radial transport and maximal spiral motion.

The logarithmic spiral arises naturally from the coexistence of linear radial growth and bounded tangential motion. The universal $1/t$ decay of the angular velocity and constraint forces follows directly from dimensional considerations and the uniform expansion.

VII. CONCLUSION

We have presented an exactly solvable model of constrained motion on a time-dependent manifold. By distinguishing between absolute and comoving velocity constraints, we demonstrated that tangential motion along an expanding boundary is generically allowed. The resulting logarithmic spiral trajectories and decaying constraint forces provide a minimal illustration of nonholonomic dynamics on evolving geometries and may serve as a useful toy model for expanding fronts and interfaces.