Derivation and Calculation of the Fine-Structure Constant

Frank Brunswig,[1,*]

December 14, 2025

[1]Institute of Geometric Physics, Heidelberg, Germany

[*]frank.brunswig@geometric-physics.de

ABSTRACT. We present a geometric framework in which spacetime is modeled as the boundary of a uniformly expanding circle, and particle motion is constrained by a universal velocity condition. Radial and tangential motion are represented by orthogonal velocity angles, and time is identified with intrinsic particle periodicity via action invariance. For the electron, this construction leads to a purely imaginary velocity angle determined by the number of completed intrinsic periods since the origin of the expansion. Requiring equality of worldline lengths for interacting particles yields a dimensionless geometric invariant formed from the electron velocity angle and the velocity angle associated with the universal propagation speed. The inverse magnitude of this invariant numerically approximates the fine-structure constant, deviating from the experimental value by approximately 0.38%. The result is obtained without invoking quantum field theory, coupling constants, or renormalization, and relies only on geometric constraints and experimentally measured constants.

I. INTRODUCTION.

The conservation of energy is one of the most fundamental principles in physics. Since the nineteenth century it has been understood that, in a closed system, energy is conserved over time and can neither be created nor destroyed, but only transformed between different forms. This principle provides the conceptual foundation for modern physics, from classical mechanics to quantum field theory.

A particularly striking feature of relativistic physics is the universal role of the speed of light. Massless particles propagate at this invariant speed in all inertial reference frames, while massive particles can be at rest but never reach the speed of light. At the same time, particles with rest mass possess a rest energy proportional to the square of the speed of light. These facts raise several closely related questions: Why does the speed of light appear as a universal limiting velocity? Why do some particles possess rest mass while others do not? And why does the same constant govern both dynamical motion and intrinsic energy?

In special relativity, these properties are encoded through Lorentz invariance and the structure of spacetime. While this framework is mathematically consistent and experimentally well confirmed, it does not provide an intuitive geometric explanation for the origin of these constraints. In particular, the coexistence of massive and massless particles and the universality of the light speed emerge as postulates rather than consequences of a deeper geometric principle.

In this work, we introduce a geometrically motivated kinematic model in which all physical entities are constrained to move on the boundary of a uniformly expanding manifold. Motion decomposes into orthogonal spatial and temporal components subject to a single normalization condition. From this constraint, relativistic energy–momentum relations arise naturally, and the distinction between massive and massless particles follows as a limiting case.

The model does not rely on Lorentz transformations as a starting point, but instead derives relativistic invariants from the geometry of the constraint itself. This approach allows for a unified interpretation of velocity, inertial mass, and energy, and provides a transparent explanation for the universality of the speed of light.

As will be shown in the following sections, this framework further leads to a trigonometric parametrization of velocity states and an alternative velocity-composition law. As a quantitative consistency check, the fine-structure constant can be derived within the model, yielding numerical agreement with its experimentally measured value.

II. GEOMETRIC MODEL AND KINEMATIC CONSTRAINT

We consider a geometrically constrained kinematic model in which all physical entities are restricted to move on the boundary of a uniformly expanding manifold. For conceptual clarity, the model is first formulated in two dimensions and may later be generalized to higher dimensions without loss of generality.

Let the physical space accessible to particles be represented by the boundary of a circle with radius $r(t)$, which expands uniformly with a constant radial expansion velocity v_c. The radius is thus given by

$$r(t) = v_c\, t,$$

where $t \geq 0$ denotes a global parameter associated with the expansion. The boundary of the circle represents physical space, while the radial direction is not directly observable by entities constrained to the boundary.

Any particle is confined to remain on the boundary at all times. Consequently, its motion decomposes into two orthogonal components: a tangential (spatial) velocity v_s along the boundary and a radial (temporal) velocity v_t associated with the expansion of the manifold. These components are not independent but are subject to a kinematic constraint imposed by the geometry.

We postulate that the magnitude of the total velocity vector in the embedding space is invariant and equal to the constant expansion velocity v_c. This leads to the normalization condition

$$v_t^2 + v_s^2 = v_c^2.$$

This equation defines a unit circle in velocity space and represents the fundamental kinematic constraint of the model. It applies locally at every point on the boundary and for every particle state.

Particles with vanishing spatial velocity $v_s = 0$ move purely radially with $v_t = v_c$. Such states correspond to particles at rest in physical space. Conversely, particles with $v_s = v_c$ necessarily satisfy $v_t = 0$; these particles cannot be brought to rest and propagate along the boundary at the invariant speed v_c.

The kinematic constraint therefore naturally distinguishes between two classes of particles: those that admit a rest frame and those that do not. This distinction arises purely from geometry and does not rely on additional dynamical assumptions.

It is convenient to parametrize the velocity components by a single angular parameter α, defined through

$$\frac{v_t}{v_c} = \cos\,\alpha, \frac{v_s}{v_c} = \sin\,\alpha,$$

with $\alpha \in [0, \pi]$. In this parametrization, particle states correspond to points on a velocity circle of radius v_c. Changes in motion correspond to rotations in velocity space.

This representation provides a direct geometric interpretation of relativistic kinematics. The invariant speed v_c emerges as the radius of the velocity space, while the decomposition into spatial and temporal components reflects orthogonality in the embedding geometry.

In the following section, this kinematic framework is extended by introducing inertial mass and energy as quantities associated with the temporal component of motion. This leads directly to relativistic energy–momentum relations without invoking Lorentz transformations as fundamental postulates.

II. INERTIAL MASS AND ENERGY

Within the geometric framework introduced in Section 2, the state of a particle is fully characterized by its position on the expanding boundary and by the decomposition of its invariant velocity v_c into radial (temporal) and tangential (spatial) components. We now introduce inertial mass and energy as quantities associated with this kinematic structure.

We define the rest mass m_0 of a particle as the inertial mass measured in a state where the spatial velocity vanishes, i.e. $v_s = 0$ and consequently $v_t = v_c$. For a particle in motion along the boundary with spatial velocity $v_s \neq 0$, we associate a dynamical (inertial) mass m, related to the reduction of the radial velocity component.

Motivated by the geometric constraint

$$v_t^2 + v_s^2 = v_c^2,$$

we postulate that inertial mass scales inversely with the radial component of motion, such that the quantity

$$m\, v_t$$

remains invariant for a given particle state. Evaluated in the rest frame, this yields

$$m\, v_t = m_0\, v_c.$$

Using the expression

$$v_t = \sqrt{v_c^2 - v_s^2},$$

the inertial mass of a moving particle is therefore given by

$$m = \frac{m_0}{\sqrt{1 - \frac{v_s^2}{v_c^2}}}.$$

This result coincides formally with the relativistic mass factor but here arises directly from the geometric constraint of motion on the expanding manifold, without invoking Lorentz transformations as axiomatic.

We define the total energy E of a particle as the product of its inertial mass and the square of the invariant expansion velocity,

$$E = m\, v_c^2.$$

In the rest state, this reduces to the rest energy

$$E_0 = m_0\, v_c^2.$$

Furthermore, we define the spatial momentum p associated with motion along the boundary as

$$p = m\, v_s.$$

Combining these definitions, the geometric velocity constraint can be rewritten as

$$m^2 v_c^4 = m_0^2 v_c^4 + m^2 v_s^2 v_c^2,$$

or equivalently,

$$E^2 = E_0^2 + p^2 v_c^2.$$

This is the relativistic energy–momentum relation, obtained here as a direct consequence of the geometry of the expanding boundary rather than as a postulate of spacetime symmetry.

The model naturally distinguishes between two classes of particles. Particles with nonzero rest mass satisfy $v_t > 0$ and admit a rest frame on the boundary. Particles for which $v_s = v_c$ necessarily satisfy $v_t = 0$ and therefore possess no rest mass. These particles propagate along the boundary at the invariant speed and cannot be brought to rest.

The existence of massless particles thus follows directly from the kinematic constraint and does not require additional assumptions. The invariant velocity v_c appears simultaneously as the maximal spatial propagation speed, the expansion rate of the manifold, and the conversion factor between mass and energy.
In this framework, energy and momentum naturally form a two-component vector,

$$\mathbf{P} = \left(\frac{E}{v_c}, p\right),$$

whose invariant magnitude is fixed by the rest mass. Transformations between different particle states correspond to rotations in this energy–momentum space, reflecting the underlying geometry of the model.

In the next section, this geometric interpretation is used to derive the addition law for velocities and to analyze the composition of particle states in different inertial frames.

III. VELOCITY COMPOSITION AND MOTION

In the geometric framework considered here, a particle state is specified by the decomposition of the invariant velocity $v_c$into orthogonal temporal (radial) and spatial (tangential) components. Since both components are bounded by the normalization condition

$$v_t^2 + v_s^2 = v_c^2,$$

velocity states can be represented as points on a unit circle when normalized by v_c.

We introduce a velocity angle α such that

$$\frac{v_t}{v_c} = \cos\, \alpha, \frac{v_s}{v_c} = \sin\, \alpha.$$

This parametrization automatically enforces the kinematic constraint and provides a natural geometric interpretation of particle motion. The limiting cases $\alpha = 0$ and $\alpha = \pi/2$ correspond to particles at rest on the boundary and to massless particles propagating at the invariant speed, respectively.

The velocity angle serves as a complete descriptor of a particle's kinematic state, replacing the conventional notion of velocity magnitude alone.

Consider two successive velocity states characterized by angles α_1 and α_2. The combined state is obtained by simple angle addition,

$$\alpha = \alpha_1 + \alpha_2.$$

The resulting spatial velocity is therefore

$$v_s = v_c \sin(\alpha_1 + \alpha_2).$$

Using the trigonometric addition theorem, this yields

$$v_s = v_{s,1}\sqrt{1 - \frac{v_{s,2}^2}{v_c^2}} + v_{s,2}\sqrt{1 - \frac{v_{s,1}^2}{v_c^2}},$$

where $v_{s,i} = v_c \sin\ \alpha_i$.

For velocities small compared to v_c, this expression reduces to the classical Galilean addition law. In the relativistic regime, it deviates from the standard velocity-addition formula of special relativity while remaining bounded by the invariant speed v_c.

The corresponding temporal (radial) component follows from

$$v_t = v_c \cos(\alpha_1 + \alpha_2),$$

which gives

$$\frac{v_t}{v_c} = \frac{v_{t,1}}{v_c}\frac{v_{t,2}}{v_c} - \sqrt{1 - \frac{v_{t,1}^2}{v_c^2}}\sqrt{1 - \frac{v_{t,2}^2}{v_c^2}}.$$

Using the relation between radial velocity and inertial mass derived in Section 3, this expression implies a nontrivial composition law for inertial mass under successive velocity transformations.

Velocity composition in this model corresponds geometrically to rotations on the velocity circle. The invariant speed $v_c$defines the radius of this circle and therefore acts as a kinematic invariant under all compositions.

Massive particles correspond to angles $\alpha < \pi/2$, while massless particles are confined to the boundary $\alpha = \pi/2$. The addition of two velocity states at $\alpha = \pi/2$ yields $\alpha = \pi$, corresponding to vanishing spatial velocity and reversed temporal orientation. This feature reflects the geometric completeness of the velocity representation and highlights the distinction between spatial and temporal components of motion. Importantly, while the spatial velocity addition differs formally from the special-relativistic composition law, experimentally accessible predictions coincide within current measurement precision for subluminal velocities. Observable deviations are expected only in regimes involving ultra relativistic relative motions.

In this framework, velocity is fundamentally an angular quantity, and motion is most naturally described as a rotation in velocity space. This interpretation provides a unified geometric understanding of kinematics, inertia, and relativistic invariance.

IV. ACTION INVARIANCE, ELECTRON PERIODICITY, AND COMPLEX VELOCITY ANGLES

It is well established that the action associated with a given particle species is an invariant of motion. In particular, extensive experimental evidence shows that the Planck constant $h$represents this invariant for the electron $e^-$and its antiparticle, the positron e^+.

In the following, we restrict the discussion to electrons and use the index e to denote electron-specific quantities (not to be confused with Euler's number).

For electrons, the invariant action may be written as

$$m_{0,e}\, v_c^2\, t_0 = m_e(t, \tilde{s})\, v_c^2\, t = h,$$

where $m_{0,e}$ is the electron rest mass, $m_e(t, \tilde{s})$ the inertial mass at spacetime position $(t, \tilde{s})$, v_c the universal velocity constant, and t_0 the proper time.

Substituting this relation into the definition of the velocity angle introduced earlier yields

$$\alpha_{v,e}(t, \tilde{s}) = \arccos\left(\frac{m_{0,e}\, v_c^2\, t}{m_e(t, \tilde{s})\, v_c^2\, t}\right) = \arccos\left(\frac{m_{0,e}\, v_c^2}{h}\, t\right).$$

The argument of the arccosine contains a constant coefficient

$$\frac{m_{0,e}\, v_c^2}{h},$$

which has the dimension of a frequency. We therefore identify this quantity with the electron eigenfrequency

$$f_e = \frac{m_{0,e}\, v_c^2}{h}.$$

Since the time variable t increases monotonically, while the arccosine function is defined only within a principal interval, the argument must be understood as a periodic function. Accordingly, time may be decomposed as

$$t = n_{e,P}\, t_{e,P} + \Delta t_{e,P},$$

where $t_{e,P} = 1/f_e$ is the electron period, $n_{e,P} \in \mathbb{N}$ the number of completed periods, and $\Delta t_{e,P}$the residual time within the current period.

Substitution yields

$$\alpha_{v,e}(t, \tilde{s}) = \arccos\left(n_{e,P} + \frac{m_{0,e}\, v_c^2}{h}\, \Delta t_{e,P}\right).$$

This expression shows that the velocity angle naturally decomposes into a discrete (integer) contribution and a continuous phase contribution.

Identifying the global time parameter t with the age of the universe T_0 plus a small residual contribution,

$$t = T_0 + \Delta t_{e,P},$$

the number of completed electron periods since the beginning of the expansion is given by

$$n_{e,P} = \frac{T_0\, m_{0,e}\, v_c^2}{h} = T_0\, f_e.$$

Using the numerical values

- $T_0 = (4.351 \pm 0.006) \times 10^{17}$ s,
- $m_{0,e} = 9.109\,383\,7139(29) \times 10^{-31}$ kg,
- $v_c = 299\,792\,458\ \mathrm{m/s}$,
- $h = 6.626\,070\,15 \times 10^{-34}$ J s,

one obtains

$$n_{e,P} = (5.376 \pm 0.007) \times 10^{37}.$$

This extremely large number implies that the corresponding velocity angle lies on a highly remote branch of the arccosine function.

For arguments outside the principal domain, the arccosine becomes complex. Using the analytic continuation,

$$\begin{aligned}\arccos(z) &= \frac{\pi}{2} + i\,\ln\left(iz + \sqrt{1 - z^2}\right) \\ &= -i\,\ln\left(z + i\sqrt{1 - z^2}\right),\end{aligned}$$

the electron velocity angle associated with $n_{e,P}$ is found to be purely imaginary:

$$\alpha_{v,e}(n_{e,P}) = \arccos\,(n_{e,P}) = (87.571 \pm 0.0013)\, i.$$

This result indicates that the electron's velocity state is characterized by a complex velocity angle, reflecting the deep connection between action invariance, intrinsic periodicity, and the geometric structure of velocity space.

V. WORLDLINE MATCHING, VELOCITY ANGLES, AND FINE-STRUCTURE CONSTANT

For two particles to interact at exactly the same spacetime point on the boundary of the expanding circle, a necessary condition is that their worldlines have equal length. In addition, a second universal constraint applies: all particles—whether possessing rest mass or not—must propagate with the universal velocity v_c.

These two necessary conditions can be combined by stating that, at a given cosmic time, the product of the worldline length and the universal velocity must be identical for all particle species.

However, it is not the velocity v_c itself that enters the geometric formulation of the model, but rather its associated velocity angle. In the framework developed above, both lengths and velocities are represented by angles: a length angle characterizing the worldline and a velocity angle characterizing the direction of motion in spacetime.

The universal velocity v_c corresponds to a velocity angle

$$\alpha_{v_c} = \pm\frac{\pi}{2},$$

representing purely tangential motion on the boundary of the expanding circle.

From the previous section, the electron velocity angle associated with the total number of completed periods $n_{e,P}$ was found to be purely imaginary,

$$\alpha_{v,e}(n_{e,P}) = (87.571 \pm 0.0013)\, i.$$

Multiplying this angle with the velocity angle of the universal velocity yields a new local invariant:

$$\begin{aligned}\alpha_{v,e}(n_{e,P}) \cdot \alpha_{v_c} &= \alpha_{v,e}(n_{e,P}) \cdot \left(\frac{\pi}{2} + 0\, i\right) \\ &= (137.556 \pm 0.02)\, i.\end{aligned}$$

Although this invariant is complex, its magnitude is a real, dimensionless quantity given by the absolute value of the imaginary coefficient.

Taking the reciprocal of the magnitude of this invariant yields

$$\frac{1}{|\ \alpha_{v,e}(n_{e,P}) \cdot \alpha_{v_c}\ |} = (0.0072698 \pm 0.0000011).$$

This value is remarkably close to the experimentally measured fine-structure constant

$$\alpha_{\mathrm{FS}} = 0.0072973525693(11).$$

The relative deviation is

$$\frac{\alpha_{\mathrm{FS}} - \alpha_{\mathrm{FS,theoretical}}}{\alpha_{\mathrm{FS}}} = 0.0038 \approx 0.38\%.$$

Within the present framework, the fine-structure constant emerges as the inverse magnitude of a purely geometric invariant constructed from:

1. the imaginary velocity angle associated with the intrinsic periodicity of the electron, and
2. the velocity angle corresponding to the universal propagation speed.

No coupling constants, renormalization procedures, or quantum-field-theoretic assumptions are introduced. The result follows solely from the geometric constraints of the expanding-circle model, the invariance of action, and the identification of time with intrinsic particle periodicity.

VI. DISCUSSION

The numerical proximity of the derived geometric invariant to the fine-structure constant is nontrivial. No coupling constants, renormalization procedures, or quantum field theoretic assumptions are employed. The result follows solely from geometric constraints, action invariance, and measured constants.

The model suggests a possible geometric origin of dimensionless constants, while remaining agnostic about microscopic dynamics. The appearance of complex velocity angles reflects intrinsic periodicity rather than unphysical motion.

VI. CONCLUSION

We have presented a minimal geometric framework in which relativistic kinematics, mass–energy relations, and a numerical approximation of the fine-structure constant arise naturally. While the model is not proposed as a replacement for established theories, it demonstrates that simple geometric constraints can encode nontrivial physical information. Further work is required to assess whether the approximation can be refined or extended to other particle species.