Tangential Motion on Expanding Analog Horizons

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ABSTRACT. We analyze the motion of point-like excitations constrained to an expanding circular front, serving as a minimal model for analog horizons. We show that the possibility of tangential motion on such an expanding horizon depends crucially on the physical interpretation of the imposed velocity constraint. While a fixed laboratory-frame speed constraint suppresses tangential degrees of freedom, a constraint formulated relative to the expanding front generically permits tangential motion. Using a nonholonomic formulation, we derive exact solutions exhibiting logarithmic spiral trajectories characterized by constant tangential speed and a time-dependent angular velocity that decays inversely with time. The associated constraint forces decrease algebraically with time. Our results clarify an ambiguity between laboratory and comoving kinematics and demonstrate that horizon-localized modes in analog gravity systems need not be purely radial.

I. INTRODUCTION.

Horizons play a central role in many areas of physics, ranging from cosmology and black hole physics to laboratory realizations in fluids, optics, and condensed matter systems. In analog gravity platforms, horizons emerge as effective boundaries for the propagation of excitations, defined by the relative motion between a medium and a characteristic signal speed. Such systems provide experimentally accessible settings in which horizon-related kinematics can be studied in controlled environments.

A common and often implicit assumption in both cosmological intuition and analog horizon models is that motion at a horizon is purely radial. In particular, excitations localized at an expanding horizon are frequently treated as having no tangential degrees of freedom, an assumption that appears natural when a fixed propagation speed is imposed. However, the precise physical meaning of this speed constraint – whether it is defined in the laboratory frame or relative to the moving horizon – often remains unstated.

In this Letter we show that this distinction is crucial. We demonstrate that tangential motion on an expanding horizon is not kinematically forbidden per se, but depends entirely on how the velocity constraint is formulated. A fixed laboratory-frame speed constraint indeed suppresses tangential motion. By contrast, a constraint imposed relative to the expanding front generically allows tangential drift along the horizon.

To make this distinction explicit, we introduce a minimal and exactly solvable model of a point-like excitation constrained to the boundary of a uniformly expanding circular front. The geometric constraint fixing the particle to the horizon is supplemented by a nonholonomic velocity constraint that enforces a fixed speed relative to the expanding front. Within this framework we derive the full equations of motion and obtain closed-form solutions.

We find that the resulting trajectories are logarithmic spirals characterized by constant tangential speed and a time-dependent angular velocity that decays inversely with time. The constraint forces required to maintain this motion decrease algebraically in time, becoming asymptotically weak despite the persistence of finite tangential drift.

Our results clarify a subtle but important ambiguity between laboratory and comoving kinematics in horizon-based models. While our analysis does not modify causal structure and does not address the cosmological horizon problem, it shows that horizon-localized modes in analog gravity systems need not be purely radial. The present work thus provides a transparent reference point for interpreting tangential degrees of freedom on expanding horizons in a wide range of analog settings.

II. MODEL AND CONSTRAINTS

We consider a point particle of mass m moving in a plane, described in polar coordinates (r, θ) by the kinetic Lagrangian

$$L = \frac{m}{2}\left(\dot{r}^{\,2} + r^2\dot{\theta}^{\,2}\right).$$

A. Geometric constraint

The particle is constrained to remain on the boundary of a circular front expanding uniformly with speed c,

$$g_1(r, t) = r - ct = 0.$$

This holonomic, explicitly time-dependent constraint fixes the particle to the instantaneous horizon.

B. Velocity constraint

We impose a second constraint on the particle's velocity, interpreted as a fixed speed relative to the expanding front rather than in the laboratory frame. The constraint takes the nonholonomic form

$$g_2(\dot{r}, r, \dot{\theta}, t) = (\dot{r} - c)^2 + (r\dot{\theta})^2 - c^2 = 0.$$

This formulation explicitly allows tangential motion along the front.

III. EQUATIONS OF MOTION

Using d'Alembert's principle with Lagrange multipliers λ_1 and λ_2 associated with g_1 and g_2, respectively, we obtain

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = \lambda_1 + 2\lambda_2(\dot{r} - c),$$

$$m(r^2\ddot{\theta} + 2r\dot{r}\,\dot{\theta}) = 2\lambda_2 r^2\dot{\theta},$$

supplemented by the constraints $g_1 = 0$ and $g_2 = 0$.

IV. EXACT SOLUITION

From the geometric constraint one immediately finds

$$r(t) = ct, \dot{r} = c.$$

Substitution into the velocity constraint yields

$$\left(r\dot{\theta}\right)^2 = c^2,$$

so that

$$\dot{\theta} = \pm\frac{1}{t}.$$

V. CONSTRAINT FORCES

Solving for the Lagrange multipliers yields

$$\lambda_2(t) = \frac{m}{2t}, \lambda_1(t) = -\frac{mc}{t}.$$

The physical constraint force

$$\mathbf{F} = F_r\,\mathbf{e}_r + F_\theta\,\mathbf{e}_\theta$$

has components

$$F_r(t) = -\frac{mc}{t}, F_\theta(t) = \pm\frac{mc}{t}.$$

Its magnitude decays algebraically as

$$|\,\mathbf{F}\,| = \frac{mc\sqrt{2}}{t}.$$

Thus, the force required to maintain tangential motion along the expanding horizon becomes asymptotically weak at late times, despite the persistence of finite tangential velocity.

Integration gives

$$\theta(t) = \theta_0 \pm \ln\left(\frac{t}{t_0}\right),$$

with constants θ_0 and $t_0 > 0$.

The trajectory in Cartesian coordinates is therefore

$$x(t) = ct\cos\left(\theta_0 \pm \ln\frac{t}{t_0}\right),$$
$$y(t) = ct\sin\left(\theta_0 \pm \ln\frac{t}{t_0}\right).$$

Eliminating t yields

$$r(\theta) = r_0\,e^{\pm(\theta-\theta_0)}, r_0 = ct_0,$$

showing that the motion follows a logarithmic spiral. The tangential speed

$$v_\theta = r\dot{\theta} = \pm c$$

is constant, while the angular velocity decays as $1/t$.

VI. IMPLICATIONS FOR ANALOG HORIZINS

Our results demonstrate that tangential motion on an expanding horizon is not forbidden by kinematics alone, but depends entirely on the frame in which the velocity constraint is defined. A laboratory-frame fixed-speed constraint suppresses tangential degrees of freedom, whereas a comoving constraint generically allows them.

This distinction is directly relevant for analog horizon systems in fluids, optics, and condensed matter, where propagation speeds are naturally defined relative to a moving medium. In such contexts, horizon-localized excitations need not be purely radial, and tangential drift may influence mode structure and angular correlations.

We emphasize that the present analysis does not modify causal structure and does not address the cosmological horizon problem. Instead, it clarifies a kinematic ambiguity that arises when motion is constrained to time-dependent horizon surfaces.

VII. CONCLUSION

We have presented an exactly solvable model of constrained motion on an expanding horizon that exposes a sharp distinction between laboratory and comoving velocity constraints. When the constraint is formulated relative to the expanding front, tangential motion is generically allowed and leads to logarithmic spiral trajectories with constant tangential speed and decaying angular velocity.

Our results provide a transparent reference framework for interpreting tangential degrees of freedom on expanding horizons and offer guidance for the modeling of horizon-localized modes in analog gravity systems.